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System of generalized resolvent equations with corresponding system of variational inclusions

Rais Ahmad · J. C. Yao

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Abstract The purpose of this paper is to introduce a new system of generalized resolvent equations with corresponding system of variational inclusions in uniformly smooth Banach spaces. We establish an equivalence relation between system of generalized resolvent equations and system of variational inclusions. The iterative algorithms for finding the approximate solutions of system of generalized resolvent equations are proposed. The convergence of approximate solutions of system of generalized resolvent equations obtained by the proposed iterative algorithm is also studied.

Keywords Generalized resolvent equations · Variational inclusions · System · Algorithm · Convergence

Mathematics Subject Classification (2002) 47H19 · 49J40

1 Introduction

In recent past, classical variational inequality has been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, elasticity and applied sciences, etc., see for example [6,12–14,17,19,21] and references therein. A useful and an important generalization of variational inequalities is a mixed variational inequality containing nonlinear term [23]. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence of a solution for the mixed variational inequalities. In 1994, Hassouni and Moudafi [10] introduced variational

R. Ahmad
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
e-mail: raisain_123@rediffmail.com

J. C. Yao (✉)
Department of Applied Mathematics, College of Sciences, National Sun Yat-Sen University, Kaohsiung 804, Taiwan, R.O.C
e-mail: yaojc@nsysu.edu.tw

inclusions which contain mixed variational inequalities as special cases. They studied the perturbed method for solving variational inclusions.

Using the concept of resolvent operator technique, Noor and Noor [18] introduced and studied resolvent equations and has established the equivalence between the mixed variational inequalities and the resolvent equations. The resolvent equations technique is being used to develop powerful and efficient numerical techniques for solving mixed (quasi) variational inequalities and related optimization problems.

In 2001, Verma [24] introduced and studied some system of variational inequalities and develop some iterative algorithms for approximating the solutions of system of variational inequalities. Pang [20], Cohen and Chaplais [9], Bianchi [7], Ansari and Yao [5] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. As generalizations of system of variational inequalities, Agarwal et al. [1] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for the system of generalized mixed quasi-variational inclusions in Hilbert spaces. Very recently, Pang and Zhu [22] considered and studied a new system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators and Lan et al. [15] studied a new system of nonlinear A-monotone multivalued variational inclusions.

Till now much attention has been drawn on the study of system of variational, quasi-variational inequalities and inclusions. The purpose of this paper is to introduce a new concept of system of generalized resolvent equations in uniformly smooth Banach spaces. We established an equivalence relation between system of generalized resolvent equations and system of variational inclusions considered by Lan et al. [15]. We have proposed number of iterative algorithms for solving system of generalized resolvent equations. The convergence criteria is also discussed.

2 Formulation and preliminaries

Throughout the paper, unless otherwise specified, we assume that E is a real Banach space with its norm $\|\cdot\|$, E^* is the topological dual of E, d is the metric induced by the norm $\|\cdot\|$, $CB(E)$ (respectively, 2^E) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(A, y)\right\}$$

where $d(x, B) = \inf_{y\in B} d(x, y)$ and $d(A, y) = \inf_{x\in A} d(x, y)$. We also assume that $\langle\cdot,\cdot\rangle$ is the duality pairing between E and E^* and $\mathcal{F}$: $E \to 2^{E^*}$ is the normalized duality mapping defined by

$$\mathcal{F}(x) = \{f \in E^*;\ \langle x,\ f\rangle = \|x\|\,\|f\| \text{ and } \|f\| = \|x\|\}, \quad \forall x \in E.$$

In the sequel, let us recall some concepts and results.

The uniform convexity of a Banach space E means that for any $\epsilon > 0$ there exists $\delta > 0$, such that for any $x, y \in E$, $\|x\| \le 1$, $\|y\| \le 1$, $\|x - y\| = \epsilon$ ensure the following inequality,

$$\|x + y\| \le 2(1 - \delta).$$

The function

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \ \|x\| = 1,\ \|y\| = 1,\ \|x-y\| = \epsilon\right\}$$

is called the modulus of the convexity of E.

The uniform smoothness of a Banach space E means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\|x+y\| + \|x-y\|}{2} - 1 \ \leq \ \epsilon\|y\|$$

holds. The function

$$\tau_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \ \|x\| = 1,\ \|y\| = t\right\}$$

is called the modulus of the smoothness of E.

We remark that the Banach space E is uniformly convex if and only if $\delta_E(\epsilon) > 0$, for all $\epsilon > 0$, and it is uniformly smooth if and only if $\lim\limits_{t\to 0} t^{-1}\tau_E(t) = 0$.

Remark 2.1 All Hilbert spaces, L_p (or l_p) spaces ($p \geq 2$) and the Sobolov spaces W_m^p ($p \geq 2$) are two-uniformly smooth, while, for $1 < p \leq 2$, L_p (or l_p) and W_m^p spaces are p-uniformly smooth.

Proposition 2.1 [5] *Let E be a uniformly smooth Banach space and $\mathcal{F}$: $E \to 2^{E^*}$ be a normalized duality mapping. Then for any $x, y \in E$,*

(*i*) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y,\ j(x+y)\rangle,\ \forall j(x+y) \in \mathcal{F}(x+y)$
(*ii*) $\langle x-y,\ j(x) - j(y)\rangle \leq 2C^2\tau_E(4\|x-y\|/C)$, *where* $C = \sqrt{\|x\|^2 + \|y\|^2/2}$.

Definition 2.1 [4]. A mapping g: $E \to E$ is said to be

(i) k-strongly accretive, $k \in (0, 1)$, if for any $x, y \in E$, there exists $j(x-y) \in \mathcal{F}(x-y)$ such that

$$\langle g(x) - g(y),\ j(x-y)\rangle \geq k\|x-y\|^2;$$

(ii) Lipschitz continuous if for any $x, y \in E$, there exists a constant $\lambda_g > 0$, such that

$$\|g(x) - g(y)\| \leq \lambda_g\|x-y\|.$$

Definition 2.2 [9] A set-valued mapping A: $E \to 2^E$ is said to be

(i) Accretive, if for any $x, y \in E$, there exists $j(x-y) \in \mathcal{F}(x-y)$, such that for all $u \in A(x)$ and $v \in A(y)$

$$\langle u - v,\ j(x-y)\rangle \geq 0;$$

(ii) k-strongly accretive, $k \in (0, 1)$, if for any $x, y \in E$, there exists $j(x-y) \in \mathcal{F}(x-y)$, such that for all $u \in A(x)$ and $v \in A(y)$,

$$\langle u - v,\ j(x-y)\rangle \geq k\|x-y\|^2;$$

(iii) m-accretive if A is accretive and $(I + \rho A)(E) = E$, for every (equivalently, for some) $\rho > 0$, where I is the identity mapping (equivalently, if A is accretive and $(I + A)(E) = E$).

Remark 2.2 [10] If $E = H$ is a Hilbert space, then $A: E \to 2^E$ is an m-accretive mapping if and only if it is a maximal monotone mapping.

Definition 2.3 Let $M: E \to 2^E$ be an m-accretive mapping. For any $\rho > 0$, the mapping $J_M^\rho: E \to E$ associated with M defined by

$$J_M^\rho(x) = (I + \rho M)^{-1}(x), \quad \forall x \in E,$$

is called the resolvent operator.

Definition 2.4 [3] The resolvent operator $J_M^\rho: E \to E$ is said to be retraction if

$$(I + \rho M)^{-1} \circ (I + \rho M)^{-1}(x) = (I + \rho M)^{-1}(x), \quad \forall x \in E.$$

Remark 2.3 It is well known that J_M^ρ is a single-valued and non-expansive mapping.

Definition 2.5 [18] A set-valued mapping $H: E \to CB(E)$ is said to be D-Lipschitz continuous if for any $x, y \in E$, there exists a constant $\lambda_{D_H} > 0$ such that

$$D(H(x),\ H(y)) \leq \lambda_{D_H} \|x - y\|.$$

Let E_1 and E_2 be two real Banach spaces, $S: E_1 \times E_2 \to E_1$, $T: E_1 \times E_2 \to E_2$, $p: E_1 \to E_1$ and $q: E_2 \to E_2$ be single-valued mappings, $H: E_1 \to 2^{E_1}$, $F: E_2 \to 2^{E_2}$ be any two multivalued mappings. Let $M: E_1 \to 2^{E_1}$ and $N: E_2 \to 2^{E_2}$ be any nonlinear mappings, $f: E_1 \to E_1$ and $g: E_2 \to E_2$ be nonlinear mappings with $f(E_1) \cap D(M) \neq \phi$ and $g(E_2) \cap D(N) \neq \phi$, respectively. Then we consider the problem of finding $(x, y) \in E_1 \times E_2$, $u \in H(x)$, $v \in F(y)$ $z' \in E_1$, $z'' \in E_2$ such that

$$\begin{aligned} S(p(x), v) + \rho^{-1} R_M^\rho(z') &= 0, \quad \rho > 0, \\ T(u, q(y)) + \gamma^{-1} R_N^\gamma(z'') &= 0, \quad \gamma > 0, \end{aligned} \tag{2.1}$$

where $R_M^\rho = I - J_M^\rho$, $R_N^\gamma = I - J_N^\gamma$ and J_M^ρ, J_N^γ are the resolvent operators associated with M and N, respectively.

The corresponding system of variational inclusions of (2.1) is the following:

Find $(x, y) \in E_1 \times E_2$, $u \in H(x)$, $v \in F(y)$ such that

$$\begin{aligned} 0 &\in S(p(x), v) + M(f(x)), \\ 0 &\in T(u, q(y)) + N(g(y)). \end{aligned} \tag{2.2}$$

Which is considered by Lan et al. [13] in Hilbert spaces.

Lemma 2.1 *$(x, y) \in E_1 \times E_2$, $u \in H(x)$, $v \in F(y)$ is a solution of system of variational inclusion (2.2) if and only if (x, y, u, v) satisfies*

$$f(x) = J_M^\rho(f(x) - \rho S(p(x), v)), \quad \rho > 0,$$

$$g(y) = J_N^\gamma(g(y) - \gamma T(u, q(y))), \quad \gamma > 0.$$

Proof The proof of Lemma 2.1 is a direct consequence of the definition of resolvent operator, and hence, is omitted. □

3 Iterative algorithms and convergence result

In this section, we first establish an equivalence relation between system of generalized resolvent equations (2.1) and system of variational inclusions (2.2). Finally, we prove the existence of a solution of (2.1) and convergence of sequences generated by the proposed algorithms.

Proposition 3.1 *The system of variational inclusions* (2.2) *has a solution* (x, y, u, v) *with* $(x, y) \in E_1 \times E_2$, $u \in H(x)$ *and* $v \in F(y)$ *if and only if system of generalized resolvent equations (2.1) has a solution* (z', z'', x, y, u, v) *with* $(x, y) \in E_1 \times E_2$, $u \in H(x)$, $v \in F(y)z' \in E_1$, $z'' \in E_2$, *where*

$$f(x) = J_M^{\rho}(z') \tag{3.1}$$

$$g(y) = J_N^{\gamma}(z'') \tag{3.2}$$

and $z' = f(x) - \rho S(p(x), v)$ *and* $z'' = g(y) - \gamma T(u, q(y))$.

Proof Let (x, y, u, v) be a solution of system of variational inclusion (2.2). Then by Lemma 2.1, it satisfies the following equations

$$f(x) = J_M^{\rho}(f(x) - \rho S(p(x), v)),$$
$$g(y) = J_N^{\gamma}(g(y) - \gamma T(u, q(y))).$$

Let $z' = f(x) - \rho S(p(x), v)$ and $z'' = g(y) - \gamma T(u, q(y))$, then we have

$$f(x) = J_M^{\rho}(z')$$
$$g(y) = J_N^{\gamma}(z''),$$

and $z' = J_M^{\rho}(z') - \rho S(p(x), v)$ and $z'' = J_N^{\gamma}(z'') - \gamma T(u, g(y))$, it follows that

$$(I - J_M^{\rho})(z') = -\rho S(p(x), v) \quad \text{and} \quad (I - J_N^{\gamma})(z'') = -\gamma T(u, q(y)),$$

i.e.

$$S(p(x), v) + \rho^{-1} R_M^{\rho}(z') = 0,$$
$$T(u, q(y)) + \gamma^{-1} R_N^{\gamma}(z'') = 0.$$

Thus, (z', z'', x, y, u, v) is a solution of system of generalized resolvent equations (2.1).

Conversely, let (z', z'', x, y, u, v) be a solution of system of generalized resolvent equations (2.1), then

$$\rho S(p(x), v) = -R_M^{\rho}(z') \tag{3.3}$$

$$\gamma T(u, q(y)) = -R_N^{\gamma}(z'') \tag{3.4}$$

Now

$$\begin{aligned}\rho S(p(x), v) &= -R_M^{\rho}(z')\\ &= -(I - J_M^{\rho})(z')\\ &= J_M^{\rho}(z') - z'\\ &= J_M^{\rho}(f(x) - \rho S(p(x), v)) - (f(x) - \rho S(p(x), v))\end{aligned}$$

which implies that

$$f(x) = J_M^{\rho}(f(x) - \rho S(p(x), v)),$$

and

$$\begin{aligned}\gamma T(u, q(y)) &= -R_N^{\gamma}(z'') \\ &= -(I - J_N^{\gamma})(z'') \\ &= J_N^{\gamma}(z'') - z'' \\ &= J_N^{\gamma}(g(y) - \gamma T(u, q(y))) - (g(y) - \gamma T(u, q(y)))\end{aligned}$$

which implies that

$$g(y) = J_N^{\gamma}(g(y) - \gamma T(u, q(y))),$$

Thus, we have

$$\begin{aligned}f(x) &= J_M^{\rho}(f(x) - \rho S(p(x), v)), \\ g(y) &= J_N^{\gamma}(g(y) - \gamma T(u, q(y))),\end{aligned}$$

thus, by Lemma 2.1, (x, y, u, v) is a solution of system of variational inclusion (2.2). □

Alternative Proof Let

$$z' = f(x) - \rho S(p(x), v) \quad \text{and} \quad z'' = g(y) - \gamma T(u, q(y)),$$

using (3.1) and (3.2), we can write

$$z' = J_M^{\rho}(z') - \rho S(p(x), v) \quad \text{and} \quad z'' = J_N^{\gamma}(z'') - \gamma T(u, q(y))$$

which implies that

$$\begin{aligned}S(p(x), v) + \rho^{-1} R_M^{\rho}(z') &= 0, \\ T(u, q(y)) + \gamma^{-1} R_N^{\gamma}(z'') &= 0,\end{aligned}$$

the required system of generalized resolvent equations.

Algorithm 3.1 For given $(x_0, y_0) \in E_1 \times E_2$, $u_0 \in H(x_0)$, $v_0 \in F(y_0)$, $z_0' \in E_1$, and $z_0'' \in E_2$, compute $\{z_n'\}$, $\{z_n''\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes,

$$f(x_n) = J_M^{\rho}(z_n') \tag{3.5}$$

$$g(y_n) = J_N^{\gamma}(z_n'') \tag{3.6}$$

$$u_n \in H(x_n) : \|u_{n+1} - u_n\| \leq D(H(x_{n+1}), H(x_n)), \tag{3.7}$$

$$v_n \in F(y_n) : \|v_{n+1} - v_n\| \leq D(F(y_{n+1}), F(y_n)), \tag{3.8}$$

$$z_{n+1}' = f(x_n) - \rho S(p(x_n), v_n) \tag{3.9}$$

$$z_{n+1}'' = g(y_n) - \gamma T(u_n, q(y_n)) \tag{3.10}$$

$n = 0, 1, 2, \ldots$

The system of generalized resolvent equations (2.1) can also be written as

$$z' = f(x) - S(p(x), v) + (I - \rho^{-1})R_M^{\rho}(z')$$
$$z'' = g(y) - T(u, q(y)) + (I - \gamma^{-1})R_N^{\gamma}(z'')$$

We use this fixed-point formulation to suggest the following iterative method.

Algorithm 3.2 For given $(x_0, y_0) \in E_1 \times E_2$, $u_0 \in H(x_0)$, $v_0 \in F(y_0)$, $z_0' \in E_1$, and $z_0'' \in E_2$, compute $\{z_n'\}$, $\{z_n''\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes,

$$f(x_n) = J_M^{\rho}(z_n')$$

$$g(y_n) = J_N^{\gamma}(z_n'')$$

$$u_n \in H(x_n) : \|u_{n+1} - u_n\| \le D(H(x_{n+1}), H(x_n)),$$

$$v_n \in F(y_n) : \|v_{n+1} - v_n\| \le D(F(x_{n+1}), F(x_n)),$$

$$z_{n+1}' = f(x_n) - S(p(x_n), v_n) + (I - \rho^{-1})R_M^{\rho}(z_n'),$$

$$z_{n+1}'' = g(y_n) - T(u_n, q(y_n)) + (I - \gamma^{-1})R_N^{\gamma}(z_n''),$$

$n = 0, 1, 2, \ldots$

For positive stepsize δ', δ'', the system of generalized resolvent equations (2.1) can also be written as

$$\begin{aligned} f(x, z') &= f(x, z') - \delta'\{z' - J_M^{\rho}(z') + \rho S(p(x), v)\} \\ &= f(x, z') - \delta'\{f(x) - J_M^{\rho}(f(x)) + \rho S(p(x), v)\}, \\ g(y, z'') &= g(y, z'') - \delta''\{z'' - J_N^{\gamma}(z'') + \gamma T(u, q(y))\} \\ &= g(y, z'') - \delta''\{g(y) - J_N^{\gamma}(g(y)) + \gamma T(u, q(y))\}. \end{aligned}$$

This fixed point formulation enables us to propose the following iterative method.

Algorithm 3.3 For given $(x_0, y_0) \in E_1 \times E_2$, $u_0 \in H(x_0)$, $v_0 \in F(y_0)$, $z_0' \in E_1$, and $z_0'' \in E_2$, compute $\{z_n'\}$, $\{z_n''\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes

$$u_n \in H(x_n) : \|u_{n+1} - u_n\| \le D(H(x_{n+1}), H(x_n)),$$

$$v_n \in F(y_n) : \|v_{n+1} - v_n\| \le D(F(y_{n+1}), F(y_n)),$$

$$f(x_{n+1}, z_{n+1}') = f(x_n, z_n') - \delta'\{f(x_n) - J_M^{\rho}(f(x_n)) + \rho S(p(x_n), v_n)\}$$

$$g(y_{n+1}, z_{n+1}'') = g(y_n, z_n'') - \delta''\{g(y_n) - J_N^{\gamma}(g(y_n)) + \gamma T(u_n, q(y_n))\}$$

$n = 0, 1, 2, \ldots$

Note that for $\delta' = \delta'' = 1$, $f(x_n, z_n') = f(x_n)$, $g(y_n, z_n'') = g(y_n)$, Algorithm 3.3 reduces to the following Algorithm which solves system of variational inclusions (2.2).

Algorithm 3.4 For given $(x_0, y_0) \in E_1 \times E_2$, $u_0 \in H(x_0)$, and $v_0 \in F(y_0)$, compute $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes,

$$f(x_{n+1}) = J_M^{\rho}(f(x_n) - \rho S(p(x_n), v_n))$$

$$g(y_{n+1}) = J_N^{\gamma}(g(y_n) - \gamma T(u_n, q(y_n)))$$

$$u_n \in H(x_n) : \|u_{n+1} - u_n\| \leq D(H(x_{n+1}), H(x_n)),$$

$$v_n \in F(y_n) : \|v_{n+1} - v_n\| \leq D(F(y_{n+1}), F(y_n)),$$

$n = 0, 1, 2, \ldots$

We now study the convergence analysis of Algorithm 3.1. In a similar way, one can study the convergence of other Algorithms.

Theorem 3.1 *Let E_1 and E_2 be two real uniformly smooth Banach spaces with module of smoothness $\tau_{E_1}(t) \leq C_1 t^2$ and $\tau_{E_2}(t) \leq C_2 t^2$ for $C_1, C_2 > 0$, respectively. Let $H\colon E_1 \to CB(E_1)$, $F\colon E_2 \to CB(E_2)$ be D-Lipschitz continuous mappings with constants λ_{D_H} and λ_{D_F}, respectively and $M\colon E_1 \to 2^{E_1}$, $N\colon E_2 \to 2^{E_2}$ be m-accretive mappings such that the resolvent operators associated with M and N are retractions. Let $f\colon E_1 \to E_1$, $g\colon E_2 \to E_2$ be both strongly accretive with constants α and β, respectively, and Lipschitz continuous with constants δ_1 and δ_2, respectively. Let $p\colon E_1 \to E_1$, $q\colon E_2 \to E_2$ be Lipschitz continuous with constants λ_p and λ_q, respectively and $S\colon E_1 \times E_2 \to E_1$, $T\colon E_1 \times E_2 \to E_2$ be Lipschitz continuous in the first and second arguments with constants λ_{S_1}, λ_{S_2} and λ_{T_1}, λ_{T_2}, respectively.*

If there exists constants $\rho > 0$ and $\gamma > 0$, such that

$$\begin{cases} 0 < \dfrac{B' + \sqrt{\theta_1} + \sqrt{\theta_4}}{1 - B'} < 1 \\ 0 < \dfrac{B'' + \sqrt{\theta_2} + \sqrt{\theta_3}}{1 - B''} < 1. \end{cases} \tag{3.11}$$

where $B' = \sqrt{1 - 2\alpha + 64 C_1 \delta_1^2}$ and $B'' = \sqrt{1 - 2\beta + 64 C_2 \delta_2^2}$, then there exists $(x, y) \in E_1 \times E_2$, $z' \in E_1$, $z'' \in E_2$, $u \in H(x)$and $v \in F(y)$ satisfying the system of generalized resolvent equations (2.1) and the iterative sequences $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.1 converge strongly to z', z'', x, y, u and v, respectively.

Proof From Algorithm 3.1, we have

$$\begin{aligned} \|z'_{n+1} - z'_n\| &= \|f(x_n) - \rho S(p(x_n), v_n) - [f(x_{n-1}) - \rho S(p(x_{n-1}), v_{n-1})]\| \\ &\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| \\ &\quad + \|x_n - x_{n-1} - \rho(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\| \end{aligned} \tag{3.12}$$

By Proposition 2.1, we have (see, for example the proof of [4, Theorem 3])

$$\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \leq (1 - 2\alpha + 64 C \delta_1^2)\|x_n - x_{n-1}\|^2 \tag{3.13}$$

Since S is Lipschitz continuous in both arguments, F is D-Lipschitz continuous and p is Lipschitz continuous, we have

$$\begin{aligned}\|S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1})\| &= \|S(p(x_n), v_n) - S(p(x_{n-1}), v_n) \\ &\quad + S(p(x_{n-1}), v_n) - S(p(x_{n-1}), v_{n-1})\| \\ &\leq \|S(p(x_n), v_n) - S(p(x_{n-1}), v_n)\| \\ &\quad + \|S(p(x_{n-1}), v_n) - S(p(x_{n-1}), v_{n-1})\| \\ &\leq \lambda_{S_1}\|p(x_n) - p(x_{n-1})\| + \lambda_{S_2}\|v_n - v_{n-1}\| \\ &\leq \lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \lambda_{S_2}D(F(y_n), F(y_{n-1})) \\ &\leq \lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|,\end{aligned} \tag{3.14}$$

Using (3.14) and Proposition 2.1, we have

$$\begin{aligned}&\|x_n - x_{n-1} - \rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\|^2 \\ &\quad \leq \|x_n - x_{n-1}\|^2 - 2\rho\langle S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}), \\ &\qquad j(x_n - x_{n-1} - \rho\{S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1})\})\rangle \\ &\quad \leq \|x_n - x_{n-1}\|^2 + 2\rho\|S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1})\| \\ &\qquad \times\|x_n - x_{n-1} - \rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\| \\ &\quad \leq \|x_n - x_{n-1}\|^2 + 2\rho(\lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|) \\ &\qquad \times\|x_n - x_{n-1} - \rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\| \\ &\quad \leq \|x_n - x_{n-1}\|^2 + \rho\lambda_{S_1}\lambda_p\{\|x_n - x_{n-1}\|^2 + \|x_n - x_{n-1} \\ &\qquad -\rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\|^2\} + \rho\lambda_{S_2}\lambda_{D_F}\{\|y_n - y_{n-1}\|^2 \\ &\qquad + \|x_n - x_{n-1} - \rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\|^2\} \\ &\quad = (1 + \rho\lambda_{S_1}\lambda_p)\|x_n - x_{n-1}\|^2 + \rho(\lambda_{S_1}\lambda_p + \lambda_{S_2}\lambda_{D_F})\|x_n - x_{n-1} \\ &\qquad -\rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\|^2 + \rho\lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|^2\end{aligned}$$

which implies that

$$\begin{aligned}&\|x_n - x_{n-1} - \rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\|^2 \\ &\quad \leq \frac{1 + \rho\lambda_{S_1}\lambda_p}{1 - \rho(\lambda_{S_1}\lambda_p + \lambda_{S_2}\lambda_{D_F})}\|x_n - x_{n-1}\|^2 \\ &\qquad + \frac{\rho\lambda_{S_2}\lambda_{D_F}}{1 - \rho(\lambda_{S_1}\lambda_p + \lambda_{S_2}\lambda_{D_F})}\|y_n - y_{n-1}\|^2 \\ &\quad = \theta_1\|x_n - x_{n-1}\|^2 + \theta_2\|y_n - y_{n-1}\|^2 \\ &\quad \leq \theta_1\|x_n - x_{n-1}\|^2 + \theta_2\|y_n - y_{n-1}\|^2 + 2\theta_1\theta_2\|x_n - x_{n-1}\|\|y_n - y_{n-1}\| \\ &\quad = (\sqrt{\theta_1}\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\|)^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\|x_n - x_{n-1} - \rho\{(S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1}))\}\| \\ &\quad \leq \sqrt{\theta_1}\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\|.\end{aligned} \tag{3.15}$$

where

$$\theta_1 = \frac{1 + \rho\lambda_{S_1}\lambda_p}{1 - \rho(\lambda_{S_1}\lambda_p + \lambda_{S_2}\lambda_{D_F})}$$

and

$$\theta_2 = \frac{\rho\lambda_{S_2}\lambda_{D_F}}{1-\rho(\lambda_{S_1}\lambda_p + \lambda_{S_2}\lambda_{D_F})}.$$

Using (3.13), (3.15), (3.12) becomes

$$\begin{aligned}\|z'_{n+1} - z'_n\| &\leq \left(\sqrt{1-2\alpha+64C\delta_1^2} + \sqrt{\theta_1}\right)\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\| \\ &= \left(B' + \sqrt{\theta_1}\right)\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\| \end{aligned} \tag{3.16}$$

where $B' = \sqrt{1-2\alpha+64C\delta_1^2}$.

Again, from Algorithm 3.1, we have

$$\begin{aligned}\|z''_{n+1} - z''_n\| &= \|g(y_n) - \gamma T(u_n, q(y_n)) - [g(y_{n-1}) - \gamma T(u_{n-1}, q(y_{n-1}))]\| \\ &\leq \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| \\ &\quad + \|y_n - y_{n-1} - \gamma(T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1})))\| \end{aligned} \tag{3.17}$$

Using the same arguments as for (3.13), we have

$$\|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|^2 \leq (1-2\beta+64C_2\delta_2^2)\|y_n - y_{n-1}\|^2 \tag{3.18}$$

Since T is Lipschitz continuous in both arguments, H is D-Lipschitz continuous and q is Lipschitz continuous, we have

$$\begin{aligned}&\|T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\| \\ &\quad = \|T(u_n, q(y_n)) - T(u_{n-1}, q(y_n)) + T(u_{n-1}, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\| \\ &\quad \leq \|T(u_n, q(y_n)) - T(u_{n-1}, q(y_n))\| + \|T(u_{n-1}, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\| \\ &\quad \leq \lambda_{T_1}\|u_n - u_{n-1}\| + \lambda_{T_2}\|q(y_n) - q(y_{n-1})\| \\ &\quad \leq \lambda_{T_1} D(H(x_n), H(x_{n-1})) + \lambda_{T_2}\lambda_q\|y_n - y_{n-1}\| \\ &\quad \leq \lambda_{T_1}\lambda_{D_H}\|x_n - x_{n-1}\| + \lambda_{T_2}\lambda_q\|y_n - y_{n-1}\| \end{aligned} \tag{3.19}$$

Using (3.19) and Proposition 2.1, we have

$$\begin{aligned}&\|y_n - y_{n-1} - \gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\|^2 \\ &\quad \leq \|y_n - y_{n-1}\|^2 - 2\gamma\langle T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1})), \\ &\qquad j(y_n - y_{n-1} - \gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\})\rangle \\ &\quad \leq \|y_n - y_{n-1}\|^2 + 2\gamma\|T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\| \\ &\qquad \times\|y_n - y_{n-1} - \gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\| \\ &\quad \leq \|y_n - y_{n-1}\|^2 + 2\gamma(\lambda_{T_1}\lambda_{D_H}\|x_n - x_{n-1}\| + \lambda_{T_2}\lambda_q\|y_n - y_{n-1}\|) \\ &\qquad \times\|y_n - y_{n-1} - \gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\| \\ &\quad \leq \|y_n - y_{n-1}\|^2 + \gamma\lambda_{T_1}\lambda_{D_H}\{\|x_n - x_{n-1}\|^2 + \|y_n - y_{n-1} \\ &\qquad -\gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\|^2\} + \gamma\lambda_{T_2}\lambda_q\{\|y_n - y_{n-1}\|^2 \\ &\qquad + \|y_n - y_{n-1} - \gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\|\}^2 \\ &\quad = (1+\gamma\lambda_{T_2}\lambda_q)\|y_n - y_{n-1}\|^2 + \gamma(\lambda_{T_1}\lambda_{D_H} + \lambda_{T_2}\lambda_q)\|y_n - y_{n-1} \\ &\qquad -\gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\|^2 + \gamma\lambda_{T_1}\lambda_{D_H}\|x_n - x_{n-1}\|^2 \end{aligned}$$

which implies that

$$\begin{aligned}
&\|y_n - y_{n-1} - \gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\|^2 \\
&\quad \le \frac{1+\gamma\lambda_{T_2}\lambda_q}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_q)}\|y_n - y_{n-1}\|^2 \\
&\qquad + \frac{\gamma\lambda_{T_1}\lambda_{D_H}}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_q)}\|x_n - x_{n-1}\|^2 \\
&\quad = \theta_3\|y_n - y_{n-1}\|^2 + \theta_4\|x_n - x_{n-1}\|^2 \\
&\quad \le \theta_3\|y_n - y_{n-1}\|^2 + \theta_4\|x_n - x_{n-1}\|^2 + 2\theta_3\theta_4\|y_n - y_{n-1}\|\|x_n - x_{n-1}\| \\
&\quad = (\sqrt{\theta_3}\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\|)^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|y_n - y_{n-1} - \gamma\{T(u_n, q(y_n)) - T(u_{n-1}, q(y_{n-1}))\}\| \\
&\quad \le \sqrt{\theta_3}\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.20}$$

where

$$\theta_3 = \frac{1+\gamma\lambda_{T_2}\lambda_q}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_q)}$$

and

$$\theta_4 = \frac{\gamma\lambda_{T_1}\lambda_{D_H}}{1-\gamma(\lambda_{T_1}\lambda_{D_H}+\lambda_{T_2}\lambda_q)}.$$

Using (3.18), (3.20), (3.17) becomes

$$\begin{aligned}
\|z''_{n+1} - z''_n\| &\le \left(\sqrt{1-2\beta+64C\delta_2^2} + \sqrt{\theta_3}\right)\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\| \\
&= \left(B'' + \sqrt{\theta_3}\right)\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\|
\end{aligned} \tag{3.21}$$

where $B'' = \sqrt{1-2\beta+64C\delta_2^2}$.

Adding (3.16) and (3.21), we have

$$\begin{aligned}
\|z'_{n+1} - z'_n\| + \|z''_{n+1} - z''_n\| \le (B' + \sqrt{\theta_1} + \sqrt{\theta_4})\|x_n - x_{n-1}\| \\
+ (B'' + \sqrt{\theta_2} + \sqrt{\theta_3})\|y_n - y_{n-1}\|
\end{aligned} \tag{3.22}$$

Also from (3.5) and (3.6), we have

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1})) + J_M^\rho(z'_n) - J_M^\rho(z'_{n-1})\| \\
&\le \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \|J_M^\rho(z'_n) - J_M^\rho(z'_{n-1})\| \\
&\le B'\|x_n - x_{n-1}\| + \|z'_n - z'_{n-1}\|
\end{aligned}$$

which implies that

$$\|x_n - x_{n-1}\| \le \frac{1}{1-B'}\|z'_n - z'_{n-1}\|, \tag{3.23}$$

and

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1})) + J_N^\gamma(z''_n) - J_N^\gamma(z''_{n-1})\| \\
&\le \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| + \|J_N^\gamma(z''_n) - J_N^\gamma(z''_{n-1})\| \\
&\le B''\|y_n - y_{n-1}\| + \|z''_n - z''_{n-1}\|
\end{aligned}$$

which implies that

$$\|y_n - y_{n-1}\| \leq \frac{1}{1 - B''}\|z''_n - z''_{n-1}\|, \tag{3.24}$$

using (3.23) and (3.24), (3.22) becomes

$$\begin{aligned}
\|z'_{n+1} - z'_n\| + \|z''_{n+1} - z''_n\| &\leq \frac{B' + \sqrt{\theta_1} + \sqrt{\theta_4}}{1 - B'}\|z'_n - z'_{n-1}\| \\
&\quad + \frac{B'' + \sqrt{\theta_2} + \sqrt{\theta_3}}{1 - B''}\|z''_n - z''_{n-1}\| \\
&\leq \theta(\|z'_n - z'_{n-1}\| + \|z''_n - z''_{n-1}\|) \qquad (3.25)
\end{aligned}$$

where

$$\theta = \max\left\{\frac{B' + \sqrt{\theta_1} + \sqrt{\theta_4}}{1 - B'}, \quad \frac{B'' + \sqrt{\theta_2} + \sqrt{\theta_3}}{1 - B''}\right\}$$

By (3.11), we know that $0 < \theta < 1$ and so (3.25) implies that $\{z'_n\}$ and $\{z''_n\}$ are both Cauchy sequences. Thus, there exists $z' \in E_1$ and $z'' \in E_2$ such that $z'_n \to z'$ and $z''_n \to z''$ as $n \to \infty$. From (3.23) and (3.24), it follows that $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences in E_1 and E_2, respectively, that is, there exists $x \in E_1$, $y \in E_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Also from (3.7) and (3.8), we have

$$\|u_{n+1} - u_n\| \leq D(H(x_{n+1}), H(x_n)) \leq \lambda_{D_H}\|x_{n+1} - x_n\|,$$

$$\|v_{n+1} - v_n\| \leq D(F(y_{n+1}), F(y_n)) \leq \lambda_{D_F}\|y_{n+1} - y_n\|,$$

and hence, $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences, let $u_n \to u$ and $v_n \to v$, respectively.

Now, we will show that $u \in H(x)$ and $v \in F(y)$. Infact, since $u_n \in H(x_n)$ and

$$\begin{aligned}
d(u_n, H(x)) &\leq \max\left\{d(u_n, H(x)), \sup_{w_1 \in H(x)} d(H(x_n), w_1)\right\} \\
&\leq \max\left\{\sup_{w_2 \in H(x_n)} d(w_2, H(x)), \sup_{w_1 \in H(x)} d(H(x_n), w_1)\right\} \\
&= D(H(x_n), H(x)),
\end{aligned}$$

we have

$$\begin{aligned}
d(u, H(x)) &\leq \|u - u_n\| + d(u_n, H(x)) \\
&\leq \|u - u_n\| + D(H(x_n), H(x)) \\
&\leq \|u - u_n\| + \lambda_{D_H}\|x_n - x\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

which implies that $d(u, H(x)) = 0$. Since $H(x) \in CB(E)$, it follows that $u \in H(x)$. Similarly, we can show that $v \in F(y)$. By continuity of $f, g, p, q, H, F, S, T, J^{\rho}_M, J^{\gamma}_N$ and Algorithm 3.1, we have

$$z' = f(x) - \rho S(p(x), v) = J^{\rho}_M(z') - \rho S(p(x), v) \in E_1$$

and

$$z'' = g(y) - \gamma T(u, q(y)) = J^{\gamma}_N(z'') - \gamma T(u, q(y)) \in E_2.$$

By Proposition (3.1), the required result follows.

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